Méthode de Gauss

* **Introduction**

Examinons les deux systèmes linéaires ci-desous :

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| **(A)** : |  | |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | | 3x | + | 5y | + | 7z | = | 101 | | 2x | + | 10y | + | 6z | = | 134 | | 1x | + | 2y | + | 3z | = | 40 | | **(B)** : |  | |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | 2x | + | 5y | + | 3z | = | 49 | **(L1)** | |  |  | 4y | + | 2z | = | 30 | **(L2)** | |  |  |  |  | 7z | = | 21 | **(L3)** | |

Le système **(A)** n'est pas très simple à résoudre car les 3 inconnues sont présentes dans les 3 équations.

Le système **(B)** est très simple à résoudre :  
**(L3)** donne : z = 3.  
Puis dans **(L2)** : 4y +6 = 30 donc y = (30-6)/4 = 6.  
Enfin dans **(L1)** : 2x + 30 + 9 = 49 donc x = (49 - 30 - 9)/2 = 5.

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| **Conclusion :** On a trouvé la solution du système **(B)** : | |  |  |  | | --- | --- | --- | | x | = | 5 | | y | = | 6 | | z | = | 3 | |

Le système linéaire **(B)** est **triangulaire supérieur**.

**Problème :** les systèmes linéaires se présentent plus souvent sous la forme du système **(A)** que sous la forme triangulaire supérieure comme le système **(B)**.

**Solution :** pour résoudre le système **(A)**, on le transforme en un système triangulaire supérieur équivalent.

**Méthode :** la méthode de Gauss se décompose en deux étapes :

**1ère Etape : élimination de Gauss :** on forme le système triangulaire supérieur équivalent en éliminant tous les termes situés sous la diagonale du système.

**2ème Etape : remontée :** on résout le système triangulaire supérieur comme on vient de le faire pour le système **(B)**.

* **1ère étape : élimination de Gauss**

Pour former un système triangulaire supérieur équivalent, on dispose de deux opérations élémentaires :

**OP 1 :** on multiple la ligne Li par un nombre réel non nul a : Li <- aLi

**OP 2 :** à la ligne Li, on ajoute b fois la ligne Lj : Li <- Li + bLj

Reprenons le système (A), à droite on l'écrit sous la forme matricielle dont l'écriture est plus légère :

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
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| |  | | --- | | **(L1)** | | **(L2)** | | **(L3)** | |  | |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | | 3x | + | 5y | + | 7z | = | 101 | | 2x | + | 10y | + | 6z | = | 134 | | 1x | + | 2y | + | 3z | = | 40 | | **forme matricielle allégée** : |  | |  |  |  |  | | --- | --- | --- | --- | | 3 | 5 | 7 | 101 | | 2 | 10 | 6 | 134 | | 1 | 2 | 3 | 40 | |  |

Itération**1 :**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
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| |  | | --- | | L1 <- L1 | | L2 <- 3 L2 - 2 L1 | | L3 <- 3 L3 - 1 L1 | |  | |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | | 3x | + | 5y | + | 7z | = | 101 | |  |  | 20y | + | 4z | = | 200 | |  |  | 1y | + | 2z | = | 19 | |  | |  | | --- | | L1 <- L1 | | L2 <- 3 L2 - 2 L1 | | L3 <- 3 L3 - 1 L1 | |  | |  |  |  |  | | --- | --- | --- | --- | | 3 | 5 | 7 | 101 | | 0 | 20 | 4 | 200 | | 0 | 1 | 2 | 19 | |  |

Itération**2 :**

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| |  | | --- | | L1 <- L1 | | L2 <- L2 | | L3 <- 20 L3 - 1 L2 | |  | |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | | 3x | + | 5y | + | 7z | = | 101 | |  |  | 20y | + | 4z | = | 200 | |  |  |  |  | 36z | = | 180 | |  | |  | | --- | | L1 <- L1 | | L2 <- L2 | | L3 <- 20 L3 - 1 L2 | |  | |  |  |  |  | | --- | --- | --- | --- | | 3 | 5 | 7 | 101 | | 0 | 20 | 4 | 200 | | 0 | 0 | 36 | 180 | |  |

**Légende :**"<-" signifie "est remplacée par",   
ainsi "L2 <- 3 L2 - 2 L1" signifie : on remplace L2 par 3 fois L2 - 2 fois L1   
ce qui permet d'éliminer x de la ligne L2.

On a obtenu un système **triangulaire supérieur** équivalent au système **(A)**.

* **2ème étape : remontée**

Le nouveau système **triangulaire supérieur** est très simple à résoudre :  
**(L3)** donne : z = 180/36=5.  
Puis dans **(L2)** : 20y +20 = 200 donc y = (200-20)/20 = 9.  
Enfin dans **(L1)** : 3x + 45 + 35 = 101 donc x = (101 - 45 - 35)/3 = 7.

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| **Conclusion :** On a trouvé la solution du système **(A)** : | |  |  |  | | --- | --- | --- | | x | = | 7 | | y | = | 9 | | z | = | 5 | |